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AN OVERLAPPING DOMAIN DECOMPOSITION METHOD WITH A VERTEX-BASED COARSE SPACE FOR RAVIART–THOMAS VECTOR FIELDS

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Abstract. In this paper, we propose a two-level overlapping domain decomposition preconditioner for three dimensional vector field problems posed in $H(\mathrm{div})$. We introduce a new coarse component, which reduces the computational complexity, associated with the coarse vertices. Numerical experiments are also presented.

1. Introduction

We consider the following vector field problem posed in $H_0(\mathrm{div};\Omega)$ in a bounded polyhedral domain Ω in $\mathbb{R}^3$: Find $\boldsymbol{u} \in H_0(\mathrm{div};\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{g},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in H_0(\mathrm{div};\Omega), \tag{1.1}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega (\alpha \,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} + \beta\,\boldsymbol{u}\cdot\boldsymbol{v})\,dx, \quad (\boldsymbol{g},\boldsymbol{v}) := \int_\Omega \boldsymbol{g}\cdot\boldsymbol{v}\,dx.$$

Here, the Hilbert space $H_0(\mathrm{div};\Omega)$ is the subspace of $H(\mathrm{div};\Omega)$, the space of vector field $\boldsymbol{u} \in \big(L^2(\Omega)\big)^3$ with $\mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)$, with vanishing normal components on the boundary. We will assume that $\boldsymbol{g} \in \big(L^2(\Omega)\big)^3$ and that $\alpha\,(\in L^\infty(\Omega))$ is nonnegative and $\beta\,(\in L^\infty(\Omega))$ is strictly positive.

The model problem introduced in (1.1) is arising from a first-order system of least-squares formulation; see [8] for more detail. We also note that fast solvers for $H(\mathrm{div})$ problems are essential in other applications, e.g., a sequential regularization method for the Navier-Stokes

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equations [19], a pseudostress-velocity formulation for the Stokes equations [9], a mixed method for Reissner–Mindlin plates [1], a mixed form for Brinkman problems [25].

Fast solution techniques for $H(\mathrm{div})$ problems are introduced in [2–6, 14–17, 20–23, 26]. Among them, overlapping type domain decomposition methods have been considered in [15, 21]. In [15], the standard way, coarse finite elements on the coarse grid, has been considered in the construction of the coarse component. However, the technique only works well for the problems with constant coefficients. With regard to the coefficients with jumps, a new type of method, also known as the generalized Dryja–Smith–Widlund (GDSW) method, has been applied in [21]. The GDSW type approach for overlapping domain decomposition methods was first introduced for solving almost incompressible elasticity problems in [11] and successfully developed for other problems, e.g., Reissner–Mindlin plate problems; see [18].

The main goal of this paper is to design the coarse component, which is an alternative to those considered in [15, 21], with less computational costs when applying the coarse component. The idea of small coarse spaces has been pioneered by Dohrmann and Widlund for the standard second order elliptic problems and compressible linear elasticity problems in [12]. Later, the method has been widely extended and adjusted; see [10, 13] and references therein. We will apply a similar method based on coarse vertices to our model problem. In case of many subdomains or multiscale problems, the size of the coarse component will be considerably lager than the sizes of the local components. This might yield a bottleneck in the parallel computation, a nature of domain decomposition methods. Thus, a coarse component with less computational complexity is essential. With this point of view, we focus on the computational cost for the coarse component not for the comprehensive algorithm.

The rest of this paper is organized as follows. In section 2, we introduce the discrete problem discretized with the Raviart–Thomas finite element. We next define the domain decomposition algorithm in section 3. Finally, we present numerical experiments in section 4.

2. The discrete problem

Let $\mathcal{T}_h$ be a triangulation of Ω into hexahedral element. We discretize the model problem (1.1) using the lowest order hexahedral Raviart–Thomas finite element conforming in $H(\mathrm{div}; \Omega)$. On each hexahedral

element $T \in \mathcal{T}_h$, the finite element has the following shape:

$$\begin{pmatrix} a_1 + b_1 x \\ a_2 + b_2 y \\ a_3 + b_3 z \end{pmatrix},$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$. Here, the six constants, a_i's and b_i's, are completely determined by degrees of freedom associated with the face f of the element T given by

$$\lambda_f(\boldsymbol{u}) := \frac{1}{|f|} \int_f \boldsymbol{u} \cdot \boldsymbol{n} \, dS.$$

For more detail, see [7, Chapter 3].

In order to obtain the discrete algebraic system, we restrict the model problem (1.1) to the finite element space. We then have following linear system:

$$Au = g, \tag{2.1}$$

where A is the stiffness matrix, u is the vector of unknowns, and g is the right-hand side vector obtained from $\boldsymbol{g}$.

We remark that our algorithms and results are similarly valid with minimal modifications for tetrahedral elements.

3. Overlapping domain decomposition method

We first divide the domain Ω into N nonoverlapping subdomains $\Omega_i, i = 1, 2, \cdots, N$. The nonoverlapping subdomains are then expanded by adding several element layers to form overlapping subdomains, $\Omega_i', i = 1, 2, \cdots, N$. We consider a two-level overlapping domain decomposition preconditioner defined by

$$M^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$$

for solving (2.1). Here, A_0 is associated with the coarse problem and A_i's are related to local subproblems defined in Ω_i'. Each action of $A_i^{-1}, i = 0, \cdots N$ can be implemented using direct methods, e.g., Cholesky factorization. Once the factorizations are available, we can apply forward and backward substituting methods in each iteration. Hence, the number of nonzero elements in the triangular systems will be important in the computation. The restriction operators R_0 and R_i's restrict the

whole problem to the coarse space and the corresponding local spaces, respectively. For more detail, see [24, Chapter 3].

3.1. Local components

The restriction operator R_i is a rectangular matrix which consists of 0 or 1 and extracts all degrees of freedom related to the extended subdomain Ω_i'. The operator A_i is defined by the following Galerkin product:

$$A_i = R_i A R_i^T.$$

We note that A_i is just a submatrix of A and we do not need any matrix multiplications since every computation can be equivalently implemented by suitable indexing.

3.2. The coarse component

We first introduce the interface Γ defined by

$$\Gamma = \left(\bigcup_{i=0}^{N} \partial\Omega_i \right) \setminus \partial\Omega.$$

The traditional GDSW methods introduced in [21] are based on the discrete harmonic extension associated with the coarse face.

For $F_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, the local stiffness matrix related to Ω_i and Ω_j is defined by

$$\begin{bmatrix} A_{II}^{(i)} & & A_{IF_{ij}}^{(i)} \\ & A_{II}^{(j)} & A_{IF_{ij}}^{(j)} \\ A_{F_{ij}I}^{(i)} & A_{F_{ij}I}^{(j)} & A_{F_{ij}F_{ij}} \end{bmatrix}.$$

We consider $u_{F_{ij}}^T = [1, 1, \cdots, 1]$ and the discrete harmonic extensions

$$u_I^{(i)} = -{A_{II}^{(i)}}^{-1} A_{IF_{ij}}^{(i)} u_{F_{ij}},$$
$$u_I^{(j)} = -{A_{II}^{(j)}}^{-1} A_{IF_{ij}}^{(j)} u_{F_{ij}}.$$

We then define u_{ij} as the extension by zero from the space of the degrees of freedom associated with $\Omega_i \cup F_{ij} \cup \Omega_j$ to that of the degrees of freedom for Ω. Then, the rows of the restriction matrix R_0 are given by all possible u_{ij}'s. The coarse matrix A_0 is then obtained by the Galerkin product

$$A_0 := R_0 A R_0^T.$$

For more detail, see [20, 21].

We now consider a vertex-based method. Let $\mathcal{F}_h$ be defined by the set of all interior faces associated with the triangulation $\mathcal{T}_h$. For a given face $f \in \mathcal{F}_h$, let us denote by c_f the center of gravity of f. We consider coarse vertices $P_1, \cdots, P_m$ and the following sets associated with the coarse vertices:

$$V_k := \left\{ f : f \in \mathcal{F}_h \wedge f \subset \Gamma, \quad \operatorname*{argmin}_{1 \le i \le m} d(c_f, P_i) = k \right\},$$

where $d(a, b)$ is the Euclidean distance between a and b. If argmin gives more than one indices, we take the smallest index. We will also denote by $\mathcal{I}_k := \left\{ k^{(1)}, \cdots, k^{(l)} \right\}$ the set of indices of subdomains that share P_k in common. Then, the local submatrix associated with P_k is given by

$$\begin{bmatrix} A_{II}^{k^{(1)}} & & & A_{IV_k}^{k^{(1)}} \\ & \ddots & & \vdots \\ & & A_{II}^{k^{(l)}} & A_{IV_k}^{k^{(l)}} \\ A_{V_kI}^{k^{(1)}} & \cdots & A_{V_kI}^{k^{(l)}} & A_{V_kV_k} \end{bmatrix}.$$

In a similar way to the face-based method, we consider the following discrete harmonic extensions with $u_{V_k}^T = [1, 1, \cdots, 1]$:

$$\begin{aligned} u_I^{k^{(1)}} &= -{A_{II}^{k^{(1)}}}^{-1} A_{IV_k}^{k^{(1)}} u_{V_k}, \\ &\cdots \\ u_I^{k^{(l)}} &= -{A_{II}^{k^{(l)}}}^{-1} A_{IV_k}^{k^{(l)}} u_{V_k}. \end{aligned}$$

Then, the construction of R_0 and A_0 is analogous to that of the face-based method.

4. Numerical experiments

We report the numerical results that illustrate the performance of the overlapping domain decomposition preconditioner in this section. We consider the domain $\Omega = (-1, 1)^3$ and decompose the domain with $4 \times 4 \times 4$ identical subdomains. For each subdomain, we use the uniform mesh consists of $8 \times 8 \times 8$ cubes. The values of α and β, which are piecewise constants, are given in a checkerboard pattern in Figure 1. We use the lowest order Raviart–Thomas finite element for the discretized problem. We solve the resulting linear system using the preconditioned conjugate gradient method. The iterations are stopped when the l^2-norm of the residual has been reduced by a factor of 10^{-6}.

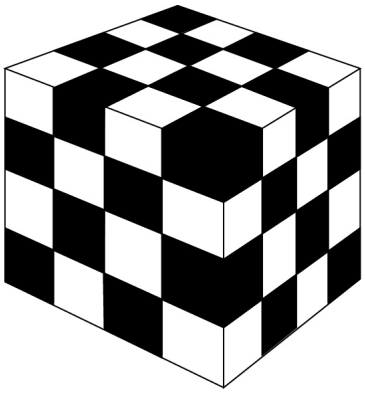

FIGURE 1. Checkerboard distribution of the coefficients.

In the first set of experiments, we fix the relative overlap H/δ and use different values of H/h with several pairs of coefficient distributions. Here, H and h are the edge lengths of subdomains and elements, respectively. The parameter δ is measured by the overlap between neighboring subdomains. The results are shown in the Tables 1 and 2. It is observed that the method is robust to jumps of the coefficients.

TABLE 1. Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{\delta} = 4$

	$\alpha_i = 100$		$\alpha_i = 10$		$\alpha_i = 1$		$\alpha_i = 0.1$		$\alpha_i = 0.01$	
H/h	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	12.02	15	18.34	18	16.61	18	9.29	16	8.27	14
4	11.89	17	15.78	19	16.53	20	9.90	18	8.32	17

TABLE 2. Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{\delta} = 4$

	$\beta_i = 100$		$\beta_i = 10$		$\beta_i = 1$		$\beta_i = 0.1$		$\beta_i = 0.01$	
H/h	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
8	12.49	19	11.19	18	16.61	18	17.37	19	16.21	18
4	12.61	21	11.26	20	16.53	20	15.54	20	16.15	20

We next fix the value H/h and vary H/δ in the second set. Other general settings are similar to those of the first set of experiments. The results are reported in Tables 3 and 4. The condition numbers and

iteration counts are insensitive to the jumps of the coefficients. Also, the condition numbers grow as H/δ increases.

TABLE 3. Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$ and $\frac{H}{h} = 8$

	$\alpha_i = 100$		$\alpha_i = 10$		$\alpha_i = 1$		$\alpha_i = 0.1$		$\alpha_i = 0.01$	
H/δ	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	21.09	17	21.24	19	20.49	20	18.72	20	14.93	18
2	12.02	15	18.34	18	16.61	18	9.29	16	8.27	14

TABLE 4. Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$ and $\frac{H}{h} = 8$

	$\beta_i = 100$		$\beta_i = 10$		$\beta_i = 1$		$\beta_i = 0.1$		$\beta_i = 0.01$	
H/δ	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	24.77	26	20.41	22	20.49	20	27.80	23	34.63	23
2	12.49	19	11.19	18	16.61	18	17.37	19	16.21	18

Finally, we compare the performance of the vertex-based GDSW coarse space with that of the traditional GDSW coarse space. As we see in the Tables 5 and 6, the condition numbers of the vertex-based method are almost twice as compared to the traditional method. Also, vertex-based method has a few more iteration counts. This is expected since the computational costs are lower when acting the coarse component in the preconditioner; see Table 7. Here, A_0 is the coarse matrix and L_0 is obtained from a Cholesky factorization of A_0. We see that the number of nonzero elements, which play a critical role in the sparse matrix computation, has been reduced substantially.

TABLE 5. Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$, $\frac{H}{h} = 8$, and $\frac{H}{\delta} = 4$

	$\alpha_i = 100$		$\alpha_i = 10$		$\alpha_i = 1$		$\alpha_i = 0.1$		$\alpha_i = 0.01$	
method	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
face-based	10.89	14	10.89	14	11.16	17	10.73	16	10.54	15
vertex-based	21.09	17	21.24	19	20.49	20	18.72	20	14.93	18

TABLE 6. Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$, $\frac{H}{h} = 8$, and $\frac{H}{\delta} = 4$

	$\beta_i = 100$		$\beta_i = 10$		$\beta_i = 1$		$\beta_i = 0.1$		$\beta_i = 0.01$	
method	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
face-based	10.72	19	11.75	18	11.16	17	12.88	16	13.69	17
vertex-based	24.77	26	20.41	22	20.49	20	27.80	23	34.63	23

TABLE 7. The numbers of nonzero elements in A_0 and L_0 for each method.

method	A_0	L_0
face-based	2160	5245
vertex-based	2197	3064

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